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The rate of convergence to the normal law in terms of pseudomoments

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Abstract We establish the rate of convergence of distributions of sums of independent identically distributed random variables to the Gaussian distribution in terms of truncated pseudomoments by implementing the idea of Yu. Studnyev for getting estimates of the rate of convergence of the order higher than $n^{-1/2}$.

Keywords Rate of convergence, truncated pseudomoments, Gaussian distribution
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1 Introduction

Applications of the central limit theorem and other weak limit theorems are closely connected to the rate of convergence to the limit distribution. The rate of convergence was studied by many authors; see [6] and the references therein. The simplest result in this direction is the Berry–Esseen inequality. Let $\{\xi_n, n \geq 1\}$ be a sequence of independent identically distributed random variables (iidrvs) with distribution function $F(x)$, $E\xi_i = 0$, and $D\xi_i = \sigma^2 < \infty$. Let $\beta_3 = \int_{\mathbb{R}} |x|^3 dF(x)$ be the absolute

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3rd moment, $\Phi_n(x) = P\{\frac{\xi_1+\xi_2+\ldots+\xi_n}{\sigma\sqrt{n}} < x\}$, and let $\Phi(x)$, $x \in \mathbb{R}$, be the standard normal distribution function. Then the Berry–Esseen inequality states that

$$\sup_{x\in\mathbb{R}}\bigl|\Phi_n(x) - \Phi(x)\bigr| \le \frac{C\beta_3}{\sigma^3\sqrt{n}}.$$

This estimate gives the rate of convergence $O(n^{-1/2})$ and the asymptotic expansions of the distribution function of the sum of iidrvs in terms of semiinvariants, presented in the book [6]. The same rate of convergence was obtained by Paulauskas [5] in terms of pseudomoments. Let $\sigma = 1$. Then the "pseudomoment" function is defined as $H(x) = F(x) - \Phi(x)$, the (absolute) third pseudomoment is defined as $\nu = \int_{\mathbb{R}} |x|^3|dH(x)|$, and we have

$$\sup_{x\in\mathbb{R}}\bigl|\Phi_n(x) - \Phi(x)\bigr| \le C\max\bigl(\nu, \nu^{\frac{1}{4}}\bigr)n^{-\frac{1}{2}}.$$

However, this rate of convergence is slow, for instance, from the point of view of financial applications. The conditions that allow one to improve the rate of convergence were formulated by several authors. After the introduction of pseudomoments in [10], they are widely used in limit theorems. Zolotarev [11] obtained very general estimates in the central limit theorem using a different type of pseudomoments. Studnyev [9] obtained the following estimate of the rate of convergence in terms of pseudomoments. Let $\{\xi_n, n \ge 1\}$ be centered iidrvs with unit variance and characteristic function $f(t)$, $\mu_k = \int_R x^k dH(x)$ be the kth-order pseudomoment, and $V(x) = V_{-\infty}^{x} H(z)$ be the variation of the function H.

Proposition 1 ([9]). *Let $F(x)$ have finite moments up to the qth order for some $q \ge 3$ and satisfy the Cramer condition $\overline{\lim}_{|t|\to\infty}|f(t)| < 1$. Then*

$$\sup_{x\in\mathbb{R}}\bigl|\Phi_n(x) - \Phi(x)\bigr| = O\left(\sum_{k=3}^{q}\frac{|\mu_k|}{n^{\frac{k-2}{2}}} + \frac{1}{n^{\frac{q-2}{2}}}\frac{1}{\sqrt{n}}\int_0^{\sqrt{n}} dx \int_{|z|>x} |z|^q dV(z)\right).$$

We can see that the condition $\mu_k = 0, 3 \le k \le r$ supplies the rate of convergence $O(n^{\frac{-r+1}{2}})$. The rate of convergence was also studied in [1, 3, 7]. In our work, we use a different type of pseudomoments and get the same rate of convergence avoiding the Cramer condition. Instead, we impose the boundedness of the truncated pseudomoments and integrability of the characteristic function.

2 Generalization of Studnyev's estimate. Main results

Let, as before, $\{\xi_n, n \ge 1\}$ be a sequence of iidrvs with $E\xi_i = 0$, $D\xi_i = \sigma^2 \in (0, \infty)$, distribution function $F(x)$, and characteristic function $f(t)$, and let $\Phi_n(x)$, $x \in \mathbb{R}$, be the distribution function of the random variable

$$S_n = (\sigma\sqrt{n})^{-1}(\xi_1 + \xi_2 + \cdots + \xi_n).$$

We assume that, for some $m \ge 3$, there exist the pseudomoments

$$\mu_k = \int_{\mathbb{R}} x^k dH(x), \quad k = 3, \ldots, m \in \mathbb{N},$$

where $H(x) = F(x\sigma) - \Phi(x)$. The truncated pseudomoments are defined as

$$\nu_n^{(1)}(m) = \int_{|x|\le\sigma\sqrt{n}} |x|^{m+1}\big|dH(x)\big|$$

("truncation from above") and

$$\nu_n^{(2)}(m) = \int_{|x|>\sigma\sqrt{n}} |x|^{m}\big|dH(x)\big|$$

("truncation from below").

Theorem 1. *Let the following conditions hold*:

(i) *The characteristic function is integrable*: $A = \int_{\mathbb{R}} |f(t)|dt < \infty$;

(ii) *The pseudomoments up to order m equal zero, and the truncated pseudomoments are bounded*:

$$\mu_k = 0, \quad k = 3, \dots, m, \text{ for some } m \ge 3, \quad \text{and}$$
$$\nu_n(m) = \max\big\{\nu_n^{(1)}(m), \nu_n^{(2)}(m)\big\} < \frac{1}{2}e^{-\frac{3}{2}}.$$

Then, for all $n \ge 2$,

$$\sup_{x\in\mathbb{R}}\big|\Phi_n(x) - \Phi(x)\big| \le 2C_m^{(1)}\frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + 2C_m^{(2)}\frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}} + \frac{\sigma A}{\pi}b^{n-1} + \nu_n(m)\frac{4e^{\frac{3}{2}}}{\pi}\frac{e^{-\frac{n}{2}}}{n},$$

where

$$C_m^{(1)} = \frac{12^{\frac{m+1}{2}}\Gamma(\frac{m+1}{2})}{\pi(m+1)!}, \qquad C_m^{(2)} = 2C_{m-1}^{(1)},$$
$$b = \exp\left\{-\frac{\pi^2}{24A^2\sigma^2(2+\pi)^2}\right\} < 1.$$

Corollary 1. *Let ξ_i be a random variable with bounded density $p(x) \le A_1$. Suppose that condition* (ii) *of Theorem* 1 *holds. Then, for all $n \ge 3$,*

$$\sup_{x\in\mathbb{R}}\big|\Phi_n(x) - \Phi(x)\big| \le 2C_m^{(1)}\frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + 2C_m^{(2)}\frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}} + 2\sigma A_1 b_1^{n-2} + \nu_n(m)\frac{4e^{\frac{3}{2}}}{\pi}\frac{e^{-\frac{n}{2}}}{n},$$

where $b_1 = \exp\{-\frac{1}{96A_1^2\sigma^2(2+\pi)^2}\} < 1$.

Note assumption (i) implies the existence of the density $p_n(x)$ of the random variable S_n. Also, let $\phi(x)$ be the density of the standard normal law.

Theorem 2. *Let the conditions of Theorem* 1 *hold. Then, for all $n \ge 2$,*

$$\sup_{x\in\mathbb{R}}\big|p_n(x) - \phi(x)\big| \le C_m^{(3)}\frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + C_m^{(4)}\frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}} + b^{n-1}\frac{\sigma\sqrt{n}}{2\pi}A + \nu_n(m)\frac{e^{\frac{3}{2}}}{\pi}\frac{e^{-\frac{n}{2}}}{n},$$

where

$$C_m^{(3)} = \frac{12^{\frac{m+2}{2}}\Gamma(\frac{m}{2}+1)}{4\pi(m+1)!}, \qquad C_m^{(4)} = 2C_{m-1}^{(3)}.$$

3 Auxiliary results. Proofs of the main results

First we prove two auxiliary results. Denote $\omega(t) = |f(\frac{t}{\sigma}) - e^{-\frac{t^2}{2}}|$.

Lemma 1. *Let $\mu_k = 0$, $k = 3, \ldots, m$. Then, for all $t \in \mathbb{R}$,*

$$\omega(t) \le \frac{|t|^{m+1}}{(m+1)!}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!}\nu_n^{(2)}(m).$$

Proof. Recall that $f(t) = \int_{-\infty}^{\infty} e^{itx} dF(x)$. Therefore, $f(\frac{t}{\sigma}) = \int_{-\infty}^{\infty} e^{\frac{itx}{\sigma}} dF(x) = \int_{-\infty}^{\infty} e^{itx} dF(x\sigma)$. By the condition of the lemma, the pseudomoments up to order m equal zero. Hence, it is easy to deduce that

$$\begin{aligned}\omega(t) &= \left| \int_{\mathbb{R}} e^{itx} dF(x\sigma) - \int_{\mathbb{R}} e^{itx} d\Phi(x) \right| = \left| \int_{\mathbb{R}} e^{itx} d\big(F(x\sigma) - \Phi(x)\big) \right| \\ &= \left| \int_{\mathbb{R}} \left(e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right) dH(x) \right| \le \int_{\mathbb{R}} \left| e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right| \big|dH(x)\big|.\end{aligned}$$

Using the inequality ([12], p. 372)

$$\left| e^{i\alpha} - 1 - \cdots - \frac{(i\alpha)^m}{m!} \right| \le \frac{2^{1-\delta}|\alpha|^{m+\delta}}{m!(m+1)^\delta}, \quad m = 0, 1, \ldots, \ \delta \in [0, 1],$$

with $\delta = 1$ and $\delta = 0$ we obtain

$$\begin{aligned}\omega(t) &\le \int_{|x|\le\sigma\sqrt{n}} \left| e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right| \big|dH(x)\big| + \int_{|x|>\sigma\sqrt{n}} \left| e^{itx} - \sum_{k=0}^{m} \frac{(itx)^k}{k!} \right| \big|dH(x)\big| \\ &\le \int_{|x|\le\sigma\sqrt{n}} \frac{|tx|^{m+1}}{(m+1)!} \big|dH(x)\big| \quad + \int_{|x|>\sigma\sqrt{n}} \frac{2|tx|^m}{m!} \big|dH(x)\big| \\ &= \frac{|t|^{m+1}}{(m+1)!}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!}\nu_n^{(2)}(m).\end{aligned}$$

The lemma is proved. □

Now, denote $T_1(n,m) = \sqrt{-2\ln(2e\nu_n(m))}$. Then, in turn, we have that $\nu_n(m) = \frac{1}{2e}\exp\{-\frac{1}{2}T_1^2(n,m)\}$. Note also that condition (ii) implies $T_1(n,m) \ge 1$.

Lemma 2. *Suppose that condition* (ii) *of Theorem* 1 *holds.*

1) *For $|t| \le T_1(n,m)$, the characteristic function allows the following bound*: $|f(\frac{t}{\sigma})| \le e^{-\frac{t^2}{6}}$.

2) *For $|t| > T_1(n,m)$, the characteristic function allows the following bound*: $|f(\frac{t}{\sigma})| \le (2e + \frac{3}{8})\nu_n(m)|t|^{m+1}$.

Proof. Evidently, $|f(\frac{t}{\sigma})| = |f(\frac{t}{\sigma}) - e^{-\frac{t^2}{2}} + e^{-\frac{t^2}{2}}| \le e^{-\frac{t^2}{2}} + \omega(t)$. Now consider two cases.

1) Let $|t| \le T_1(n,m)$. Then we can deduce from Lemma 1 that

$$\begin{aligned}
\left|f\left(\frac{t}{\sigma}\right)\right| &\le e^{-\frac{t^2}{4}}\left(e^{-\frac{t^2}{4}} + e^{\frac{t^2}{4}}\omega(t)\right) \\
&\le e^{-\frac{t^2}{4}}\left(1 + e^{\frac{t^2}{4}}\left(\frac{|t|^{m+1}}{(m+1)!}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!}\nu_n^{(2)}(m)\right)\right) \\
&\le e^{-\frac{t^2}{4}}\left(1 + e^{\frac{T_1^2(n,m)}{4}}t^2\left(\frac{T_1^{m-1}(n,m)}{(m+1)!}\nu_n^{(1)}(m) + \frac{2T_1^{m-2}(n,m)}{m!}\nu_n^{(2)}(m)\right)\right) \\
&\le e^{-\frac{t^2}{4}}\left(1 + t^2 e^{\frac{T_1^2(n,m)}{4}}\nu_n(m)\left(\frac{T_1^{m-1}(n,m)}{(m+1)!} + \frac{2T_1^{m-2}(n,m)}{m!}\right)\right) \\
&= e^{-\frac{t^2}{4}}\left(1 + \frac{1}{2e}t^2 e^{-\frac{T_1^2(n,m)}{4}}\left(\frac{T_1^{m-1}(n,m)}{(m+1)!} + \frac{2T_1^{m-2}(n,m)}{m!}\right)\right). \qquad (1)
\end{aligned}$$

Consider the function $f_1(x) = \exp\{-\frac{x^2}{4}\}x^{m-1}$. It attains its maximal value at the point $x = \sqrt{2(m-1)}$, and this value equals

$$f_{1,\max} = \exp\left\{-\frac{m-1}{2}\right\}\left(2(m-1)\right)^{\frac{m-1}{2}}.$$

Furthermore,

$$\begin{aligned}
\exp\left\{-\frac{m-1}{2}\right\}\frac{(2(m-1))^{\frac{m-1}{2}}}{(m+1)!} &\le \frac{\exp\{-\frac{m-1}{2}\}(2(m-1))^{\frac{m-1}{2}}}{m(m+1)\sqrt{2\pi(m-1)}(m-1)^{m-1}e^{-(m-1)}} \\
&= \left(\frac{2e}{m-1}\right)^{\frac{m-1}{2}}\frac{1}{\sqrt{2\pi(m-1)}}\frac{1}{m(m+1)} \\
&\le \frac{1}{m(m+1)}.
\end{aligned}$$

The last fraction attains its maximal value at the point $m = 3$. Therefore,

$$\exp\left\{-\frac{T_1^2(n,m)}{4}\right\}\frac{T_1^{m-1}(n,m)}{(m+1)!} \le \frac{1}{12}.$$

Similarly,

$$\exp\left\{-\frac{T_1^2(n,m)}{4}\right\}\frac{2T_1^{m-2}(n,m)}{m!} \le \frac{1}{3}.$$

From (1) together with two last bounds it follows that

$$\begin{aligned}
\left|f\left(\frac{t}{\sigma}\right)\right| &\le e^{-\frac{t^2}{4}}\left(1 + \frac{1}{2e}t^2\left(\frac{1}{12} + \frac{1}{3}\right)\right) \le e^{-\frac{t^2}{4}}\left(1 + \frac{1}{12}t^2\right) \\
&\le e^{-\frac{t^2}{4}}e^{\frac{t^2}{12}} \le e^{-\frac{t^2}{6}},
\end{aligned}$$

and the proof of the first statement follows.

2) Now, let $|t| > T_1(n,m)$. Then we get from Lemma 1 that

$$\begin{aligned}\left|f\left(\frac{t}{\sigma}\right)\right| &\le e^{-\frac{t^2}{2}} + \omega(t)\\ &\le e^{-\frac{T_1^2(n,m)}{2}} + \frac{|t|^{m+1}}{(m+1)!}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!}\nu_n^{(2)}(m)\\ &\le \nu_n(m)\left(2e + \frac{|t|^{m+1}}{(m+1)!} + \frac{2|t|^m}{m!}\right).\end{aligned} \tag{2}$$

Recall that $T_1(n,m) > 1$. Then $|t| > T_1(n,m) > 1$, and from (2) we get that

$$\left|f\left(\frac{t}{\sigma}\right)\right| \le \nu_n(m)\left(2e|t|^{m+1} + \frac{|t|^{m+1}}{24} + \frac{2|t|^{m+1}}{6}\right) \le \left(2e + \frac{3}{8}\right)\nu_n(m)|t|^{m+1},$$

whence the proof follows. □

Now we are in position to prove the main results.

Proof of Theorem 1. Let F and G be two distribution functions with characteristic functions f and g, respectively, and suppose that G has a density function, which we denote G'. We shall use the following inequality from [4], p. 297:

$$\sup_{x\in\mathbb{R}}\left|F(x) - G(x)\right| \le \frac{2}{\pi}\int_0^T \left|f(t) - g(t)\right|\frac{dt}{t} + \frac{24\sup|G^{'}|}{\pi T}.$$

Taking $F(x) = \Phi_n(x)$ and $G(x) = \Phi(x)$, we have

$$\rho_n := \sup_{x\in\mathbb{R}}\left|\Phi_n(x) - \Phi(x)\right| \le \frac{2}{\pi}\int_0^T \left|f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}}\right|\frac{dt}{t} + \frac{24}{\pi\sqrt{2\pi}T}. \tag{3}$$

Let $n \ge 2$. First, from the elementary inequality

$$\left|u^n - v^n\right| \le |u - v|\sum_{k=1}^n |u|^{k-1}|v|^{n-k}$$

and from Lemma 1 it follows that, for $t \le T_1(n,m)\sqrt{n}$,

$$\begin{aligned}\left|f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}}\right| &\le \omega\left(\frac{t}{\sqrt{n}}\right)\sum_{k=1}^n \left|f\left(\frac{t}{\sigma\sqrt{n}}\right)\right|^{k-1} e^{-\frac{t^2}{2n}(n-k)}\\ &\le \omega\left(\frac{t}{\sqrt{n}}\right)\sum_{k=1}^n e^{-\frac{t^2}{6}\frac{n-1}{n}} \le \omega\left(\frac{t}{\sqrt{n}}\right) n e^{-\frac{t^2}{12}}\\ &\le n\left(\frac{|t|^{m+1}}{(m+1)!n^{\frac{m+1}{2}}}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!n^{\frac{m}{2}}}\nu_n^{(2)}(m)\right)\exp\left\{-\frac{t^2}{12}\right\}\\ &= \exp\left\{-\frac{t^2}{12}\right\}\left(\frac{|t|^{m+1}}{(m+1)!n^{\frac{m-1}{2}}}\nu_n^{(1)}(m) + \frac{2|t|^m}{m!n^{\frac{m-2}{2}}}\nu_n^{(2)}(m)\right).\end{aligned} \tag{4}$$

Second, introduce the following notation:

$$C_{m,n}^{(1)} = \frac{12^{\frac{m-1}{2}} \Gamma(\frac{m+1}{2})}{2n^{\frac{m-1}{2}}(m+1)!}, \qquad C_{m,n}^{(2)} = 2C_{m-1,n}^{(1)},$$

$$T_2(n,m) = \frac{1}{\sqrt{2\pi}(C_{m,n}^{(1)}\nu_n^{(1)}(m) + C_{m,n}^{(2)}\nu_n^{(2)}(m))}.$$

Then

$$\frac{24}{\pi\sqrt{2\pi}T_2(n,m)} = C_m^{(1)} \frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + C_m^{(2)} \frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}}. \tag{5}$$

Let $T_3(n,m) = (T_1(n,m)\sqrt{n}) \wedge T_2(n,m)$. Then it follows from (3) and (5) that

$$\begin{aligned}
\rho_n &\le \frac{2}{\pi}\int_0^{T_3(n,m)} \left| f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}} \right| \frac{dt}{t} + \frac{2}{\pi}\int_{T_3(n,m)}^{T_2(n,m)} \left| f\left(\frac{t}{\sigma\sqrt{n}}\right)\right|^n \frac{dt}{t} \\
&\quad + \frac{2}{\pi}\int_{T_3(n,m)}^{T_2(n,m)} e^{-\frac{t^2}{2}} \frac{dt}{t} + \frac{24}{\pi\sqrt{2\pi}T_2(n,m)} \\
&= I_1(n,m) + I_2(n,m) + I_3(n,m) + C_m^{(1)} \frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + C_m^{(2)} \frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}}.
\end{aligned} \tag{6}$$

Since $T_3(n,m) \le T_1(n,m)\sqrt{n}$, from (4) we get that

$$\begin{aligned}
I_1(n,m) &= \frac{2}{\pi}\int_0^{T_3(n,m)} \left| f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}} \right| \frac{dt}{t} \\
&\le \frac{2}{\pi}\int_0^{T_3(n,m)} \left(\frac{t^m}{(m+1)!n^{\frac{m-1}{2}}}\nu_n^{(1)}(m) + \frac{2t^{m-1}}{m!n^{\frac{m-2}{2}}}\nu_n^{(2)}(m) \right) e^{-\frac{t^2}{12}} dt \\
&\le \frac{12^{\frac{m+1}{2}}\Gamma(\frac{m+1}{2})}{\pi(m+1)!n^{\frac{m-1}{2}}}\nu_n^{(1)}(m) + \frac{2\cdot 12^{\frac{m}{2}}\Gamma(\frac{m}{2})}{\pi m! n^{\frac{m-2}{2}}}\nu_n^{(2)}(m) \\
&= C_m^{(1)} \frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + C_m^{(2)} \frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}}.
\end{aligned} \tag{7}$$

If $T_3(n,m) = T_2(n,m)$, then $I_2(n,m) = 0$ and $I_3(n,m) = 0$. Therefore, we consider the case $T_3(n,m) = T_1(n,m)\sqrt{n}$. Then

$$I_2(n,m) = \frac{2}{\pi}\int_{T_3(n,m)}^{T_2(n,m)} \left| f\left(\frac{t}{\sigma\sqrt{n}}\right)\right|^n \frac{dt}{t} = \frac{2}{\pi}\int_{T_1(n,m)/\sigma}^{T_2(n,m)/\sigma\sqrt{n}} \left|f(t)\right|^n \frac{dt}{t}.$$

Now we apply the following result of Statulevičius [8]: if a random variable with characteristic function $f(t)$ has a density $p(x) \le d < \infty$ and variance σ^2, then, for any $t \in \mathbb{R}$,

$$\left|f(t)\right| \le \exp\left\{ -\frac{t^2}{96d^2(2\sigma|t| + \pi)^2} \right\}. \tag{8}$$

It follows from condition (i) that the density $p(x)$ of any ξ_n can be obtained as the inverse Fourier transform $p(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-itx} f(t)dt$ and $p(x) \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(t)|dt = \frac{A}{2\pi}$. Besides, the function $\frac{t^2}{(2\sigma t+\pi)^2}$ is increasing for $t > 0$. Therefore, for $|t| \ge$

$T_1(n,m)/\sigma$ (recall that $T_1(n,m) > 1$),

$$\big|f(t)\big| \le \exp\left\{-\frac{\pi^2}{24A^2\sigma^2(2+\pi)^2}\right\} =: b,$$

and $0 < b < 1$. Then

$$I_2(n,m) = \frac{2}{\pi}\int_{T_1(n,m)/\sigma}^{T_2(n,m)/\sigma\sqrt{n}} \big|f(t)\big|^n \frac{dt}{t} \le \frac{2\sigma}{\pi} b^{n-1} \int_0^\infty \big|f(t)\big| dt = \frac{\sigma A}{\pi} b^{n-1}. \quad (9)$$

Finally, we bound $I_3(n,m)$. Note that $I_3(n,m)$ is nonzero only if $T_1(n,m)\sqrt{n} < T_2(n,m)$. Therefore,

$$\begin{aligned} I_3(n,m) &\le \frac{2}{\pi}\int_{T_1(n,m)\sqrt{n}}^{\infty} e^{-\frac{t^2}{2}} \frac{dt}{t} \le \frac{2}{\pi}\frac{e^{-\frac{nT_1^2(n,m)}{2}}}{nT_1^2(n,m)} \\ &\le \frac{2(2e\nu_n(m))^n}{\pi n} \le \frac{4e\nu_n(m)}{\pi n}\big(2e\nu_n(m)\big)^{n-1} \le \nu_n(m)\frac{4e \cdot e^{-\frac{n-1}{2}}}{\pi n} \\ &= \nu_n(m)\frac{4e^{\frac{3}{2}}}{\pi}\frac{e^{-\frac{n}{2}}}{n}. \end{aligned} \quad (10)$$

Relations (6)–(10) supply the proof of Theorem 1. □

Remark 1. Let the following conditions hold: $\mu_k = 0$, $k = 3, \dots, m$, $m \ge 3$. Then

$$\begin{aligned} \sup_{x\in\mathbb{R}}\big|\Phi_1(x) - \Phi(x)\big| &= \sup_{x\in\mathbb{R}}\big|F(x\sigma) - \Phi(x)\big| \\ &\le \left(\frac{6}{\pi(m+1)!} + \frac{2}{\pi\sqrt{2\pi}}\right)\max\Big(\nu_1(m), \big(\nu_1(m)\big)^{\frac{1}{m+2}}\Big). \end{aligned}$$

Indeed, let $n = 1$. Theorem is obvious when $\nu_1(m) > 1$. Let $\nu_1(m) \le 1$. Put $T = (\nu_1(m))^{-\frac{1}{m+2}}$ into *(3)*. Then from Lemma 1 it follows that

$$\begin{aligned} \rho_1 &= \sup_{x\in\mathbb{R}}\big|\Phi_1(x) - \Phi(x)\big| = \sup_{x\in\mathbb{R}}\big|F(x\sigma) - \Phi(x)\big| \\ &\le \frac{2}{\pi}\int_0^T \left(\frac{|t|^{m+1}}{(m+1)!}\nu_1^{(1)}(m) + \frac{2|t|^m}{m!}\nu_1^{(2)}(m)\right)\frac{dt}{t} + \frac{24}{\pi\sqrt{2\pi}T} \\ &\le \frac{2}{\pi}\left(\frac{T^{m+1}}{(m+1)\cdot(m+1)!}\nu_1^{(1)}(m) + \frac{2T^m}{m\cdot m!}\nu_1^{(2)}(m)\right) + \frac{24}{\pi\sqrt{2\pi}}\big(\nu_1(m)\big)^{\frac{1}{m+2}} \\ &\le \big(\nu_1(m)\big)^{\frac{1}{m+2}}\left(\frac{2}{\pi}\frac{3}{(m+1)!} + \frac{24}{\pi\sqrt{2\pi}}\right). \end{aligned}$$

Proof of Corollary 1. Proof is similar to that of Theorem 1. We apply inequality (8) and recall again that the function $\frac{t^2}{(2\sigma t+\pi)^2}$ is increasing for $t > 0$. Therefore, for $|t| \ge T_1(n,m)/\sigma$ (recall that $T_1(n,m) > 1$),

$$\big|f(t)\big| \le \exp\left\{-\frac{1}{96A_1^2\sigma^2(2+\pi)^2}\right\} =: b_1,$$

and $0 < b_1 < 1$. It follows from [2], p. 510, that $\int_{-\infty}^{\infty} |f(t)|^2 dt \le 2\pi A_1$. Therefore,

$$I_2(n,m) = \frac{2}{\pi} \int_{T_1(n,m)/\sigma}^{T_2(n,m)/\sigma\sqrt{n}} \bigl|f(t)\bigr|^n \frac{dt}{t} \le \frac{2\sigma}{\pi} b_1^{n-2} \int_0^{\infty} \bigl|f(t)\bigr|^2 dt = 2\sigma A_1 b_1^{n-2}.$$

Corollary 1 is proved. □

Remark 2. For $n = 2$, we can get estimates similar to those in Remark 1.

Proof of Theorem 2. As it was mentioned before, condition (i) implies the existence of a density for the random variable ξ_k, so the random variable S_n has the density

$$p_n(x) = \int_{-\infty}^{\infty} e^{-itx} f^n\left(\frac{t}{\sigma\sqrt{n}}\right) dt.$$

Since $\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ is the density of the standard normal law, we have $\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} dt$ and

$$\begin{aligned} \bigl|p_n(x) - \phi(x)\bigr| &= \frac{1}{2\pi} \left| \int_{-\infty}^{\infty} e^{-itx} f^n\left(\frac{t}{\sigma\sqrt{n}}\right) dt - \int_{-\infty}^{\infty} e^{-itx} e^{-\frac{t^2}{2}} dt \right| \\ &\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}} \right| dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \bigl|p_n(x) - \phi(x)\bigr| \le{}& \frac{1}{2\pi} \int_{|t| \le T_1(n,m)\sqrt{n}} \left| f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}} \right| dt \\ &+ \frac{1}{2\pi} \int_{|t| > T_1(n,m)\sqrt{n}} \left| f\left(\frac{t}{\sigma\sqrt{n}}\right) \right|^n dt \\ &+ \frac{1}{2\pi} \int_{|t| > T_1(n,m)\sqrt{n}} e^{-\frac{t^2}{2}} dt \\ ={}& I_1 + I_2 + I_3. \end{aligned} \tag{11}$$

From the conditions of the theorem, Lemmas 1 and 2, and from (4) $(n \ge 2)$ we obtain the following: for $|t| \le T_1(n,m)\sqrt{n}$ and $\nu_n(m) < \frac{1}{2} e^{-\frac{3}{2}}$,

$$\begin{aligned} I_1 &= \frac{1}{2\pi} \int_{|t| \le T_1(n,m)\sqrt{n}} \left| f^n\left(\frac{t}{\sigma\sqrt{n}}\right) - e^{-\frac{t^2}{2}} \right| dt \\ &\le \frac{1}{2\pi} \int_{|t| \le T_1(n,m)\sqrt{n}} \left(\frac{|t|^{m+1} \nu_n^{(1)}(m)}{(m+1)! n^{\frac{m-1}{2}}} + \frac{2|t|^m \nu_n^{(2)}(m)}{m! n^{\frac{m}{2}-1}} \right) e^{-\frac{t^2}{12}} dt \\ &\le \frac{12^{\frac{m+2}{2}} \Gamma(\frac{m}{2}+1)}{4\pi(m+1)!} \frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + \frac{2 \cdot 12^{\frac{m+1}{2}} \Gamma(\frac{m-1}{2})}{4\pi m!} \frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}} \\ &= C_m^{(3)} \frac{\nu_n^{(1)}(m)}{n^{\frac{m-1}{2}}} + C_m^{(4)} \frac{\nu_n^{(2)}(m)}{n^{\frac{m-2}{2}}}. \end{aligned} \tag{12}$$

From the conditions of the theorem, similarly to (9), we get

$$\begin{aligned} I_2 &= \frac{1}{2\pi}\int_{|t|>T_1(n,m)\sqrt{n}}\left|f\left(\frac{t}{\sigma\sqrt{n}}\right)\right|^n dt \\ &= \frac{\sigma\sqrt{n}}{2\pi}\int_{|z|>T_1(n,m)/\sigma}\left|f(z)\right|^n dz \le b^{n-1}\frac{\sigma\sqrt{n}}{2\pi}A. \end{aligned} \tag{13}$$

$$\begin{aligned} I_3 &= \frac{1}{2\pi}\int_{|t|>T_1(n,m)\sqrt{n}} e^{-\frac{t^2}{2}}dt \le \frac{1}{2\pi}\int_{T_1(n,m)\sqrt{n}}^{\infty} e^{-\frac{t^2}{2}}dt \\ &\le \frac{e^{-\frac{nT_1^2(n,m)}{2}}}{2\pi\sqrt{n}T_1(n,m)} \le \frac{(2e\nu_n(m))^n}{2\pi\sqrt{n}} \\ &\le \frac{e\nu_n(m)}{\pi\sqrt{n}}e^{-\frac{n-1}{2}} = \nu_n(m)\frac{e^{\frac{3}{2}}}{\pi}\frac{e^{-\frac{n}{2}}}{\sqrt{n}}. \end{aligned} \tag{14}$$

Relations (11)–(14) supply the proof of Theorem 2. □

4 Example

We give an example of application of Theorem 1. It is similar to the example of [12], p. 375, where the discrete distribution was considered. Define the distribution function $F(x)$ as

$$F(x) = \begin{cases} \Phi(x) & \text{if } |x| \ge \epsilon, \\ \Phi(-\epsilon) & \text{if } -\epsilon < x < -\theta\epsilon, \\ \Phi(\epsilon) & \text{if } \theta\epsilon < x < \epsilon, \\ \frac{1}{2} + \frac{\Phi(\epsilon)-\frac{1}{2}}{\theta\epsilon}x & \text{if } |x| \le \theta\epsilon, \end{cases}$$

where $0 < \epsilon < 1$, and $0 < \theta < 1$ is the root of the equation

$$\int_0^{\epsilon} x^2 d\Phi(x) = \frac{(\theta\epsilon)^2}{3}\left(\Phi(\epsilon) - \frac{1}{2}\right). \tag{15}$$

This equation has a unique solution because $\int_0^{\epsilon} x^2 d\Phi(x) \le \frac{\epsilon^2}{3}(\Phi(\epsilon) - \frac{1}{2})$. Indeed, on one hand,

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{x^2}{2} + \frac{x^4}{2^2 2!} - \frac{x^6}{2^3 3!} + \frac{x^8}{2^4 4!} - \cdots\right),$$

and therefore,

$$\Phi(\epsilon) - \frac{1}{2} = \int_0^{\epsilon}\varphi(x)dx = \frac{1}{\sqrt{2\pi}}\left(\epsilon - \frac{\epsilon^3}{6} + \frac{\epsilon^5}{40} - \frac{\epsilon^7}{7\cdot 2^3 3!} + \cdots\right).$$

So

$$\Phi(\epsilon) - \frac{1}{2} \ge \frac{1}{\sqrt{2\pi}}\left(\epsilon - \frac{\epsilon^3}{6}\right).$$

On the other hand,

$$\begin{aligned}
\int_0^{\epsilon} x^2 d\Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_0^{\epsilon} \left(x^2 - \frac{x^4}{2} + \frac{x^6}{2^2 2!} - \frac{x^8}{2^3 3!} + \cdots \right) dx \\
&= \frac{1}{\sqrt{2\pi}} \left(\frac{\epsilon^3}{3} - \frac{\epsilon^5}{10} + \frac{\epsilon^7}{7 \cdot 2^2 2!} - \frac{\epsilon^9}{9 \cdot 2^3 3!} + \cdots \right) \\
&\le \frac{1}{\sqrt{2\pi}} \left(\frac{\epsilon^3}{3} - \frac{\epsilon^5}{10} + \frac{\epsilon^7}{56} \right) \\
&\le \frac{1}{\sqrt{2\pi}} \left(\frac{\epsilon^3}{3} - \frac{\epsilon^5}{10} + \frac{\epsilon^5}{56} \right) = \frac{1}{\sqrt{2\pi}} \epsilon^3 \left(\frac{1}{3} - \frac{23}{280} \epsilon^2 \right) \\
&= \frac{1}{\sqrt{2\pi}} \frac{\epsilon^3}{3} \left(1 - \frac{69}{280} \epsilon^2 \right) \le \frac{1}{\sqrt{2\pi}} \frac{\epsilon^3}{3} \left(1 - \frac{\epsilon^2}{6} \right) \le \frac{\epsilon^2}{3} \left(\Phi(\epsilon) - \frac{1}{2} \right),
\end{aligned}$$

and we immediately get that

$$\int_0^{\epsilon} x^2 d\Phi(x) \le \frac{\epsilon^2}{3} \left(\Phi(\epsilon) - \frac{1}{2} \right).$$

It is obvious that the density function is bounded. Moreover, $F(x)$ is symmetric. Therefore, $\mu_1 = 0$ and $\mu_3 = 0$. Furthermore, taking into account (15), consider

$$\begin{aligned}
\sigma^2 &= \int_{-\infty}^{\infty} x^2 dF(x) = \int_{|x| \ge \epsilon} x^2 d\Phi(x) + \int_{-\theta\epsilon}^{\theta\epsilon} x^2 \frac{\Phi(\epsilon) - \frac{1}{2}}{\theta\epsilon} dx \\
&= \int_{|x| \ge \epsilon} x^2 d\Phi(x) + \frac{\Phi(\epsilon) - \frac{1}{2}}{\theta\epsilon} \frac{2}{3} (\theta\epsilon)^3 = \int_{-\infty}^{\infty} x^2 d\Phi(x) = 1.
\end{aligned}$$

This means that $\mu_2 = 0$. Consider further the pseudomoments

$$\begin{aligned}
\nu_4 &= \int_{-\infty}^{\infty} x^4 \left| d\big(F(x) - \Phi(x)\big) \right| = \int_{-\epsilon}^{\epsilon} x^4 \left| d\big(F(x) - \Phi(x)\big) \right| \\
&\le \epsilon^4 \int_{-\epsilon}^{\epsilon} \left| d\big(F(x) - \Phi(x)\big) \right| \le \epsilon^4 4 \left(\Phi(\epsilon) - \frac{1}{2} \right), \\
\nu_n^{(1)}(3) &= \int_{|x| \le \sigma\sqrt{n}} x^4 \left| d\big(F(x) - \Phi(x)\big) \right| = \int_{-\epsilon}^{\epsilon} x^4 \left| d\big(F(x) - \Phi(x)\big) \right|,
\end{aligned}$$

where ϵ can be chosen so that $\nu_4 \le \frac{1}{2} e^{-\frac{3}{2}}$ and $\nu_n^{(1)}(3) \le \frac{1}{2} e^{-\frac{3}{2}}$. Then

$$\nu_n^{(2)}(3) = \int_{|x| > \sigma\sqrt{n}} |x|^3 \left| d\big(F(x) - \Phi(x)\big) \right| = 0.$$

Hence, condition (ii) of Theorem 1 holds. Therefore, the function $F(x)$ satisfies the conditions of Theorem 1 with $m = 3$.

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